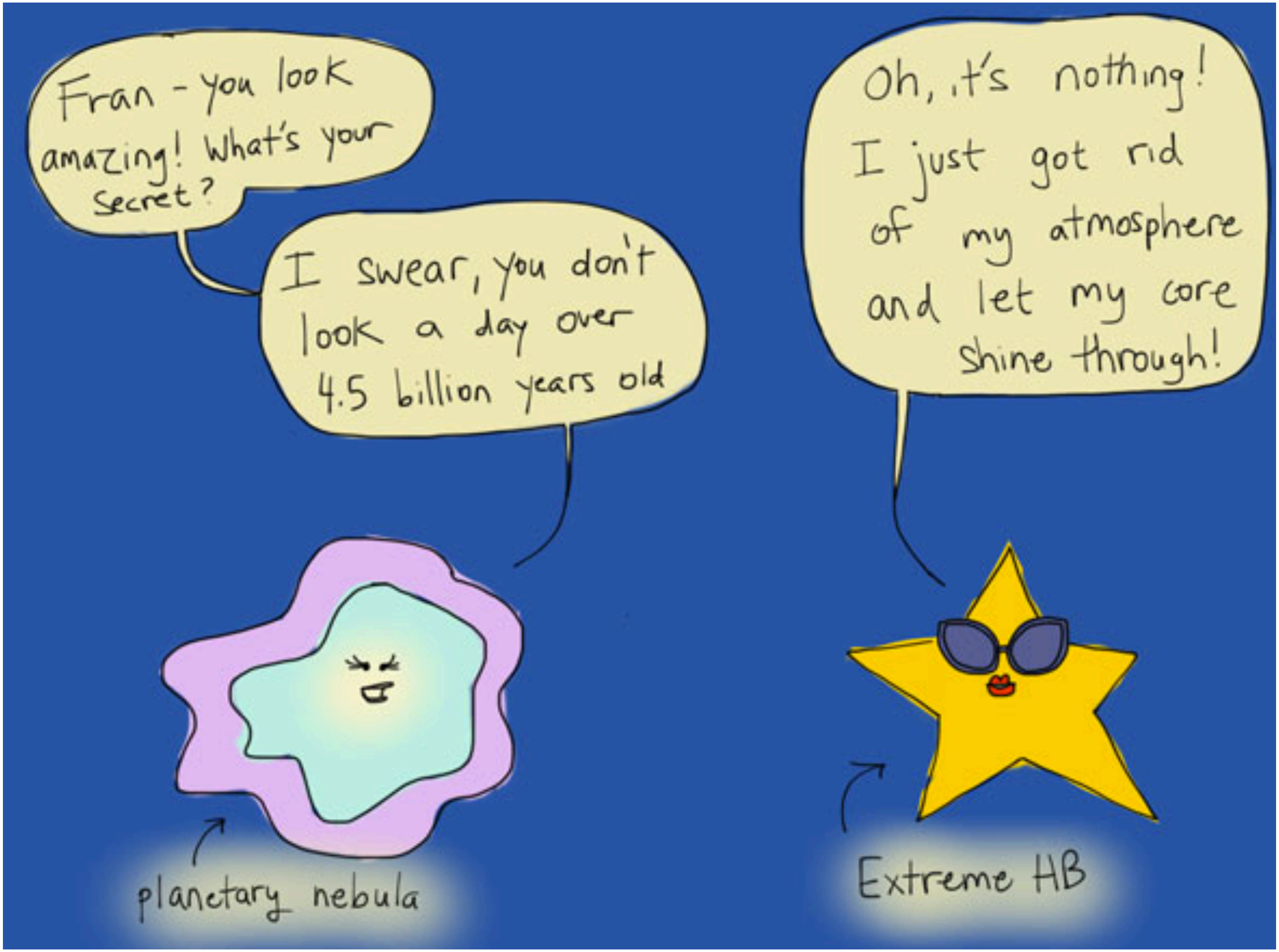
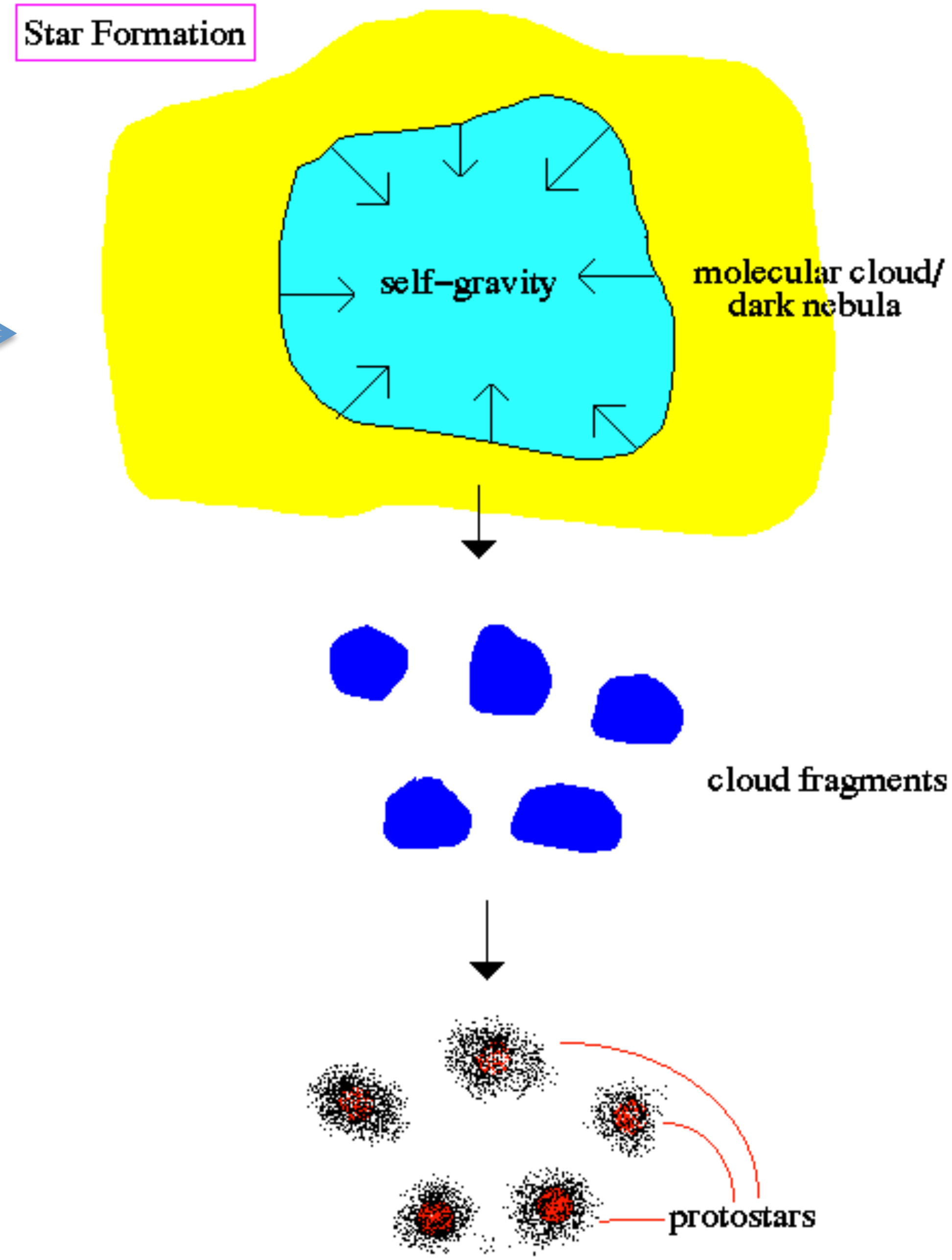

Fran - you look amazing! What's your secret?
I swear, you don't look a day over 4.5 billion years old
Oh, it's nothing! I just got rid of my atmosphere and let my core shine through!
planetary nebula
Extreme HB

Outline



Star Formation
self-gravity
molecular cloud/
dark nebula
cloud fragments
protostars

The Stability of Clouds: Jeans Instability

We start with the basic equations for fluid motions:

The continuity equation (conservation of mass):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

where $\mathbf{v}$ is a vector.

The 2nd equation is the momentum equation, which basically states $F = ma$:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P - \rho \nabla \phi \tag{2}$$

Note that the equation for hydrostatic equilibrium can be derived from Euler's in the case equation when $\mathbf{v}$ is set to 0.

Finally, the gravitational potential ϕ is given by

$$\nabla^2 \phi = 4\pi G \rho \tag{3}$$

The Stability of Clouds: Jeans Instability

Consider an infinite, static medium with a density of ρ. Now, consider a small perturbation to this medium:

$$\mathbf{v} = \mathbf{v_0} + \delta\mathbf{v},\ \ \rho = \rho_0 + \delta\rho,\ \ \phi = \phi_0 + \delta\phi \tag{9}$$

If we subsititute these values into the continuity equation we get:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{10}$$

goes to

$$\frac{\partial\delta\rho}{\partial t} + \delta\mathbf{v}\cdot\nabla(\rho_0) + \rho_0\nabla\cdot\delta\mathbf{v} = 0 \tag{11}$$

where it is assumed $\mathbf{v_0} = 0$ and that all 2nd order terms, i.e. $\delta\rho\delta v$, go to 0. For the momentum equation we get:

The Stability of Clouds: Jeans Instability

For the momentum equation we get:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\phi \tag{12}$$

goes to

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c_s^2}{\rho}\nabla\delta\rho - \nabla\delta\phi \tag{13}$$

where the $(\mathbf{v} \cdot \nabla)\mathbf{v}$ term is 2nd order and vanishes and c_s is the sound speed for an ideal, isothermal gas which is equal to

$$c_s = \sqrt{\frac{kT}{\mu m_H}} \tag{14}$$

(where $P = c_s^2\rho$ is the ideal gas law) and we have assumed that the unperturbed gas is in hydrostatic equilibrium, i.e.

$$\frac{c_s^2}{\rho_0}\nabla\rho_0 = -\nabla\phi_0 \tag{15}$$

The Stability of Clouds: Jeans Instability

Now we combine equations 6 and 8 by taking the derivative of equation 6 with respect to time:

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \frac{\partial \delta \mathbf{v}}{\partial t} \cdot \nabla(\rho_0) + \rho_0 \nabla \cdot \frac{\partial \delta \mathbf{v}}{\partial t} = 0 \tag{16}$$

which if we assume ρ_0 is constant (and thus $\nabla \rho_0 = 0$) we get

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \delta \mathbf{v}}{\partial t} \tag{17}$$

and then substituting equation 8 into the new equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \nabla^2 \delta\rho + \nabla^2 \delta\phi \tag{18}$$

Finally, using the Poisson equation, we arrive at our final equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \nabla^2 \delta\rho + 4\pi G \delta\rho \right) \tag{19}$$

The Stability of Clouds: Jeans Instability

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Note that the equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \nabla^2 \delta\rho \right) \tag{20}$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \nabla^2 \delta\rho \tag{21}$$

is the wave equation for a pressure wave with sound speed c_s.

The Stability of Clouds: Jeans Instability

Let's rewrite this as 1-D equation (we can assume 1-D perturbations).

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \frac{\partial^2 \delta\rho}{\partial x^2} + 4\pi G \delta\rho \right) \tag{22}$$

We adopt the following solution:

$$\delta\rho = \delta\rho_0 e^{i(wt-kx)} \tag{23}$$

If we plug this solution into the problem, the we then get the following dispersion equation:

$$w^2 = c_s^2(k^2 - k_j^2) \tag{24}$$

where

$$k_j^2 = 4\pi G \frac{\rho_0}{c_s^2} \tag{25}$$

The Stability of Clouds: Jeans Instability

$$k_j^2 = 4\pi G \frac{\rho_0}{c_s^2} \tag{25}$$

Thus, if $k > k_J$, w is real and we get a normal wave equation. However, if $k < k_J$, then the solution for w is imaginary. The resulting time dependence solution for the perturbation density is

$$\delta\rho = \frac{\delta\rho_0}{2}\left(e^{(|k^2 - k_J^2|)^{1/2}t} + e^{-(|k^2 - k_J^2|)^{1/2}t}\right) \tag{26}$$

which increases exponentially with time. In other words, gravity wins over pressure, and the pertubations are unstable. We can convert that into a length, the Jeans length

$$k_J = 2\pi/\lambda_J, \;\; \lambda_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}} \tag{27}$$

the Jeans length can be turned into a mass

$$m_j = \rho_0 \lambda_J^3 = \frac{c_s^3 \pi^{3/2}}{G^{3/2}\rho_0^{1/2}} = \left(\frac{\pi k T}{\mu m_H G}\right)^{3/2} \rho_0^{-1/2} \tag{28}$$

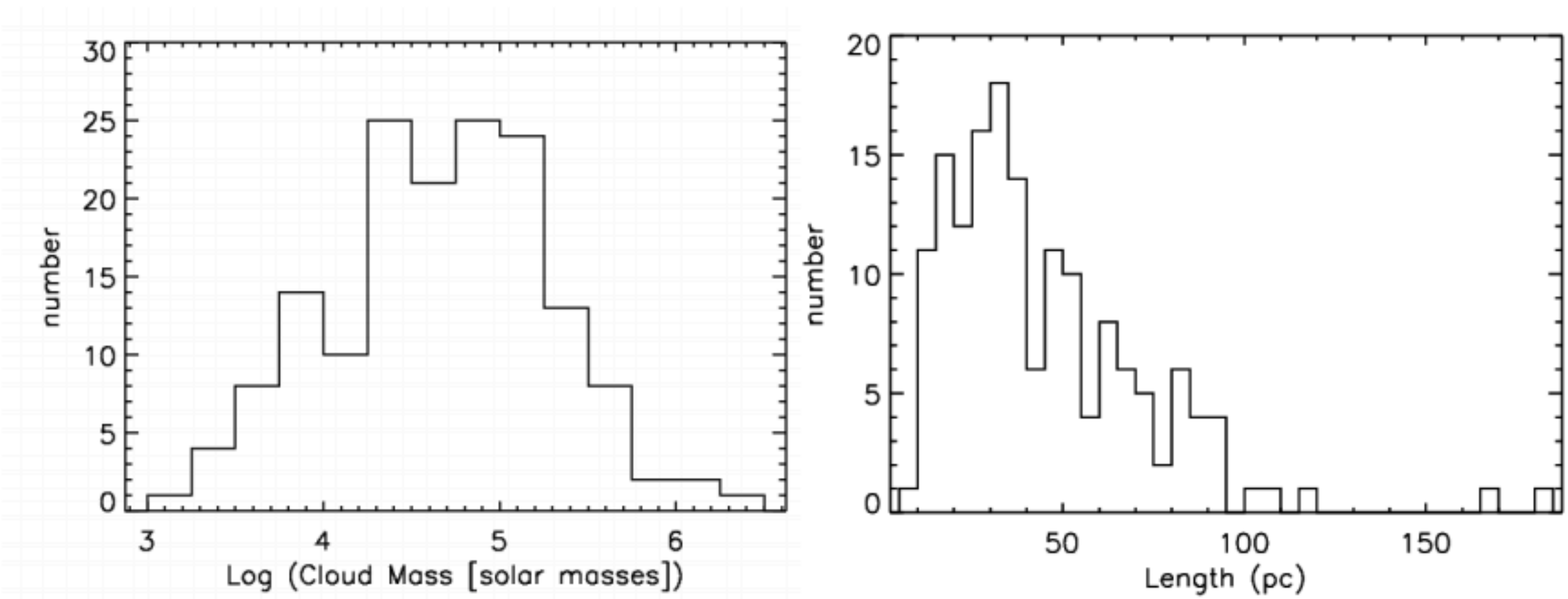
Are Clouds stable to Gravitational Collapse ?

For $n(H_2) = 40\ cm^{-3}$ $\lambda_J = 6$ pc $M_J = 450$ Msun (cloud size)

For $n(H_2) = 100\ cm^{-3}$ $\lambda_J = 4.$ pc $M_J = 350$ Msun (cloud size)

For $n(H_2) = 1000\ cm^{-3}$ $\lambda_J = 1.2$ pc $M_J = 100$ Msun (clump size)

Cloud masses and lengths exceed Jeans length masses.

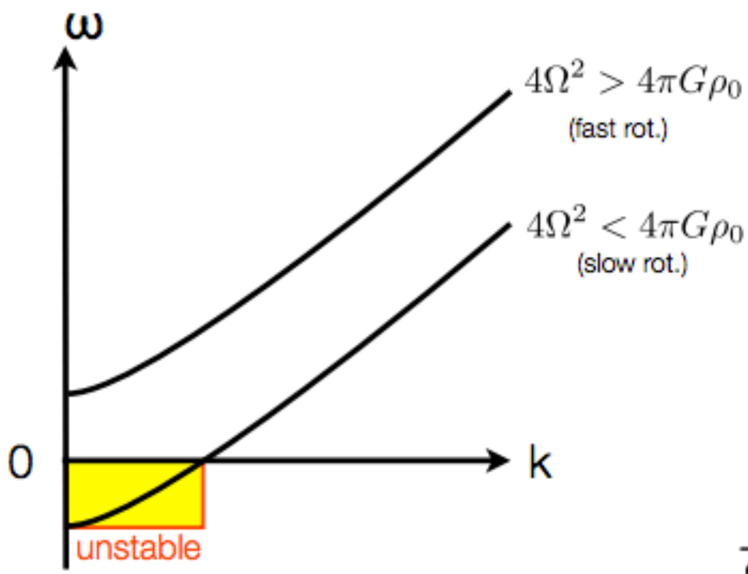


From clouds in the inner galaxy from Heyer et al. 2009

Jeans instability with rotation

The second case is the important special case where θ=90°, i.e. where the rotation axis and the perturbation are perpendicular to each other. The dispersion relation takes the simple form:

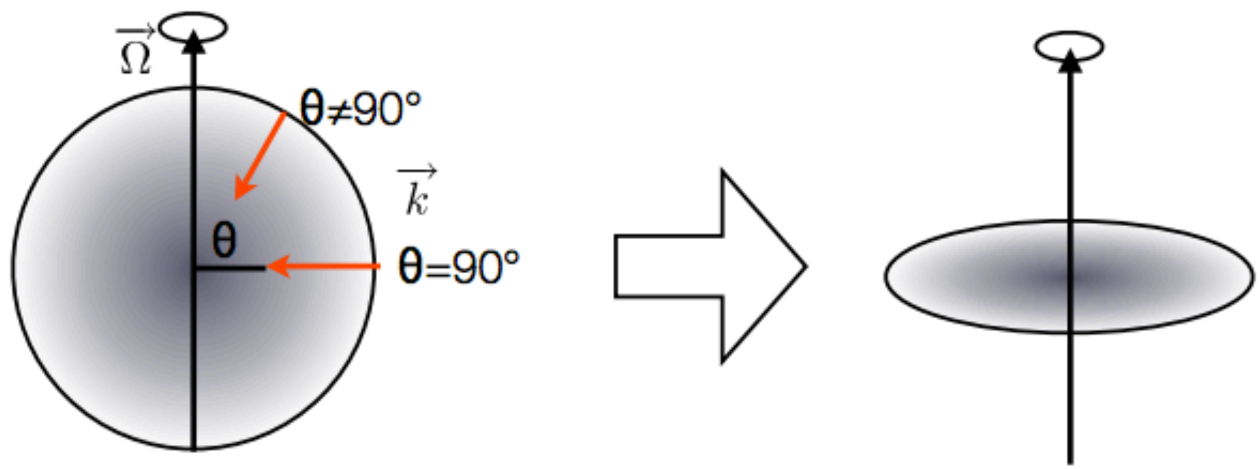
$$\omega^2 = 4\Omega^2 + c^2k^2 - 4\pi G\rho_0$$



Clearly, the additional rotation term acts stabilizing. For sufficiently large rotation, in the direction perpendicular to the rotation axis, rotation can stabilize the largest scales from gravitational collapse. This is nothing else than the fact that angular momentum is an enemy of star formation, and that instead of collapsing into a point, the cloud must now collapse into a disk.

small wavelength (large k): stabilized by pressure

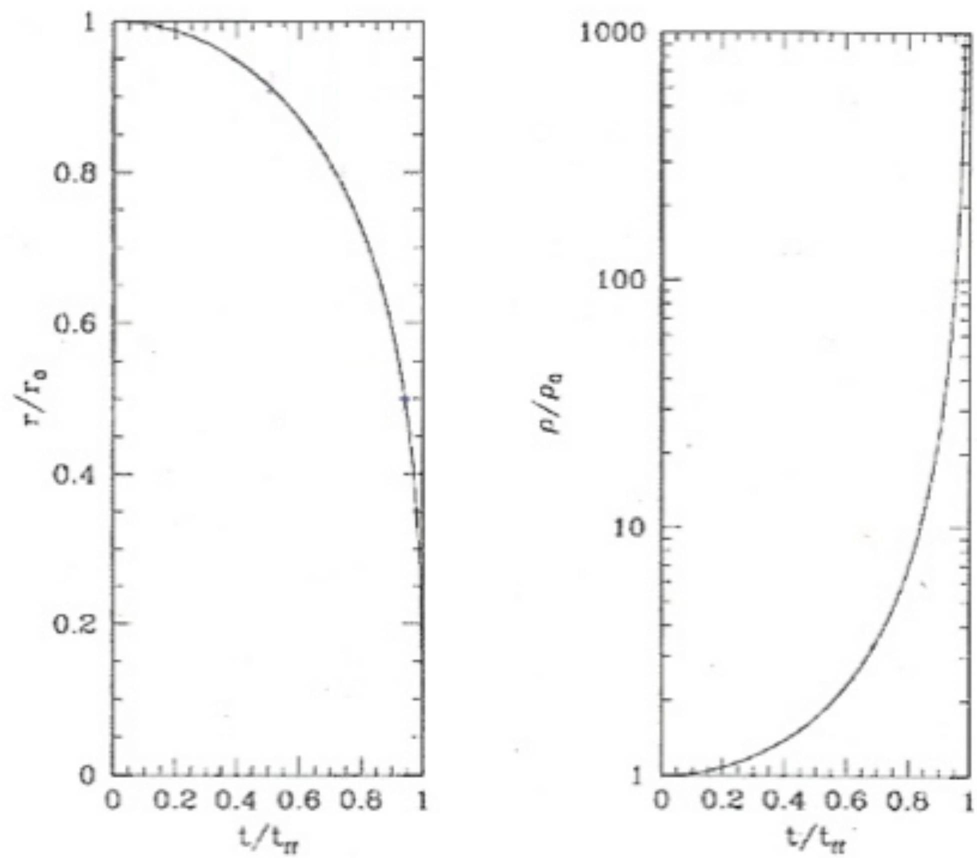
large wavelength (small k): stabilized by rotation



Collapse of a homogeneous Gas Cloud

- Imagine P=0 (correct at least at the beginning), the same as T=0
- Most of the action takes place at the last minute
  - Since $t_{ff} \propto 1/\sqrt{\rho}$
- Initially r and ρ change slowly then very rapidly:

For example, $\frac{r}{r_0} = 0.5$ only when $\frac{t}{t_{ff}} \approx 0.95$

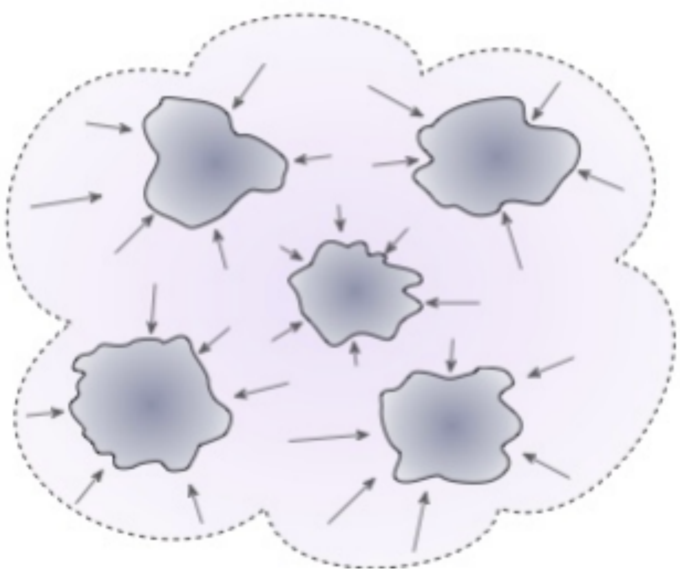


Fragmentation

During the collapse, **ρ** increases. As long as the density still remains adequately low for the cloud to be transparent, the released thermal energy is radiated into the universe and the temperature remains approximately constant. As

$$M_J \propto \sqrt{\frac{T^3}{\rho}}$$

suggests, this leads to a decrease of the Jeans mass. In particular, sub-sections of the cloud suddenly surpass their own Jeans limit and start collapsing on their own. As also t_{ff} is smaller for higher densities, these sub-collapses proceed faster. This clearly leads to fragmentation.



Opacity Limit

- We have neglected up to now the effect of pressure, or equivalently assumed a low, constant temperature. As the density becomes higher and higher, the gas increasingly becomes opaque to its own radiation. Therefore, the temperature must start to rise at some point. This eventually leads to a minimum fragment size into which the cloud can break up. This is know as the opacity limit.
- In the isothermal case:

$$M_J \propto \sqrt{\frac{T^3}{\rho}}$$

- In the opaque limit (e.g. adiabatic), $T \sim P^{2/5}$ (since $p \sim \rho T$) -> $T \sim \rho^{2/3}$

$$M_J \propto T^{3/2} \rho^{-1/2} \propto \rho^{1/2}$$

- So in the adiabatic case the Jeans mass increases with density, which means that the combination of decrease of M_J in the isothermal regime and increase of it in the adiabatic leads to a minimum fragment mass

Minimum Fragment Mass

We can estimate the critical mass by some simple energy considerations.

1)Heating

The gravitational binding energy of a collapsing gas ball is

$$W = -q\frac{GM^2}{R}$$

q=3/5 for cst. density

The collapse happens on a typical timescale t_{ff}

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$

The liberated binding energy per second which heats the gas is therefore

$$\frac{|W|}{t_{ff}} = q\frac{2\sqrt{2}}{\pi}\left(\frac{G^3M^5}{R^5}\right)^{1/2}$$

2) Cooling

In the same time cools the gas by radiating as a blackbody at the surface of the gas blob.

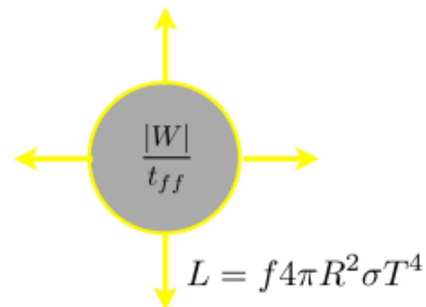
$$L = f4\pi R^2\sigma T^4$$

f: correction factor i.e
radiation efficiency<1

Minimum Fragment Mass

The gas will start to heat up (become adiabatic) as soon as the cooling becomes slower than the heating. In order to prevent this we must have:

$$\frac{|W|}{t_{ff}} \lesssim L$$



i.e.

$$q \frac{2\sqrt{2}}{\pi} \left(\frac{G^3 M^5}{R^5} \right)^{1/2} \lesssim f 4\pi R^2 \sigma T^4$$

Solving for M gives the criterion

$$M^5 \lesssim M_{crit}^5 = \frac{2\pi^4 \sigma^2}{6^3 q^2} f^2 R^9 T^8$$

In the same time, the mass must be bigger than the Jeans mass for collapse to proceed, so

$$M_J \lesssim M \lesssim M_{crit}$$

When M_J and M_{crit} become equal, we hit the opacity limit.

Minimum Fragment Mass

Using our earlier result for the Jeans mass $\lambda_J = \sqrt{\frac{45}{16\pi}} \frac{c}{\sqrt{G\rho}}$ $M_J = \frac{4}{3}\pi\rho\lambda_J^3$

and combining the equations above, we finally find for the minimal fragment mass

$$M_{frag} = \frac{225\,15^{1/4}}{16\pi\,2^{3/4}} \left(\frac{q}{\sigma G^3}\right)^{1/2} \frac{c^{9/2}}{f^{1/2}T^2}$$

(=5.24)

Recalling that the isothermal sound speed is $c^2 = \frac{k}{\mu m_H} T$ we get using q=3/5

$$M_{frag} \approx 1.13 \times 10^{31} \frac{T^{1/4}}{f^{1/2}} \quad \text{[g]}$$

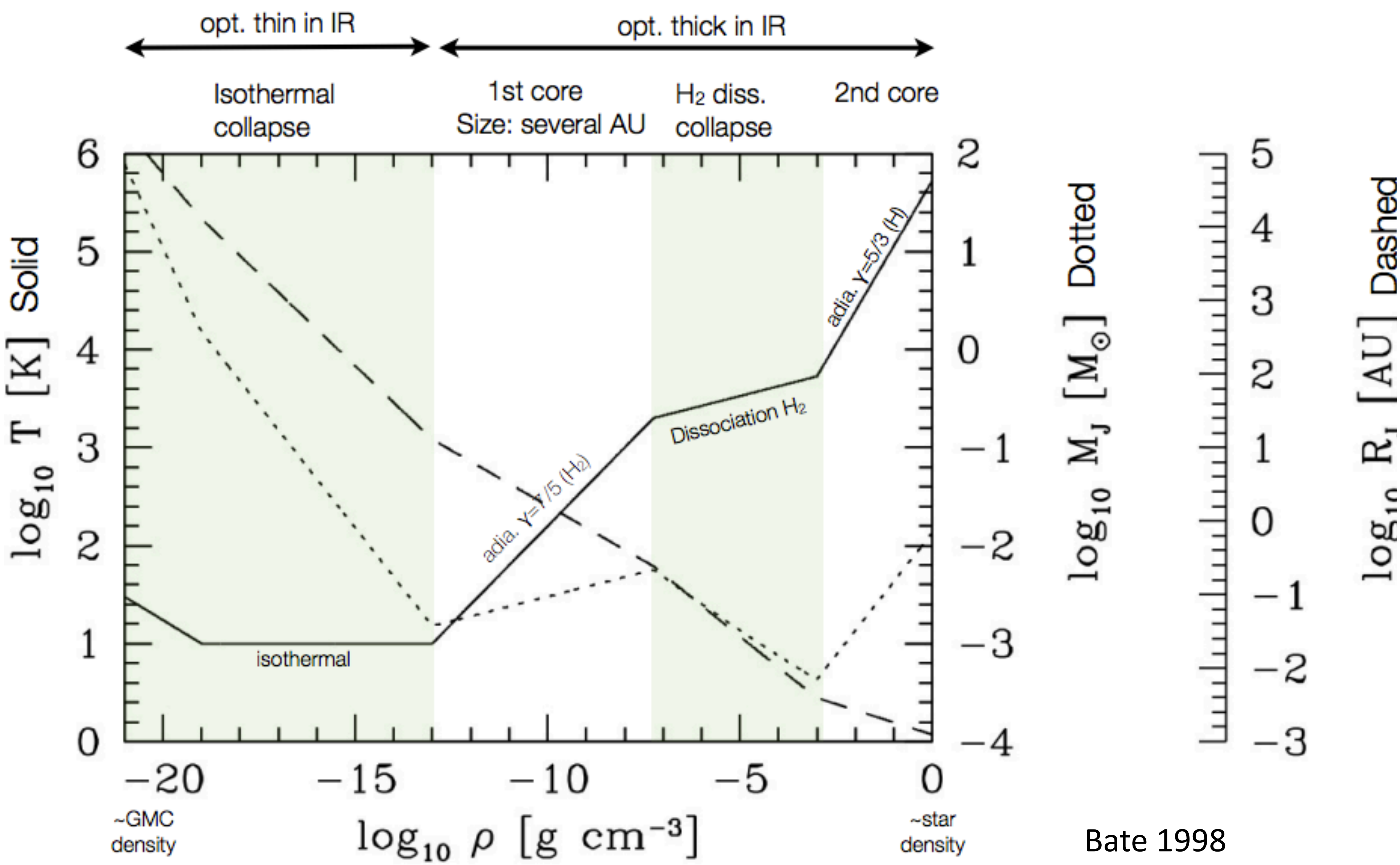
We see that the result only depends weakly on the temperature. Numerically we get

T= 10 K, f=1 => $M_{frag} \approx 0.01 M_\odot \approx 10 M_{Jupiter}$

T= 100 K, f=0.01 => $M_{frag} \approx 0.17 M_\odot$

From this very simple estimate we thus see that cloud fragmentation leads to objects going from the upper planetary mass domain to low mass stars (M dwarfs, the most frequent type of stars in the galaxy).

Collapse



Bate 1998

At Last

- https://www.youtube.com/watch?v=YbdwTwB8jtc

Reading Assignment

- These notes
- Chapter 2 and 10 in ‘The Formation of Stars’ (Stahler & Palla)
- Paper by Bade 1998:
  - http://adsabs.harvard.edu/abs/1998ApJ...508L..95B

- Review by Larson: http://www.astro.yale.edu/larson/papers/Ringberg93.pdf